Tikrit university
College of Engineering
Mechanical Engineering Department

Lectures on Engineering Analysis

Chapter 3
Laplace Transforms

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Laplace Transform

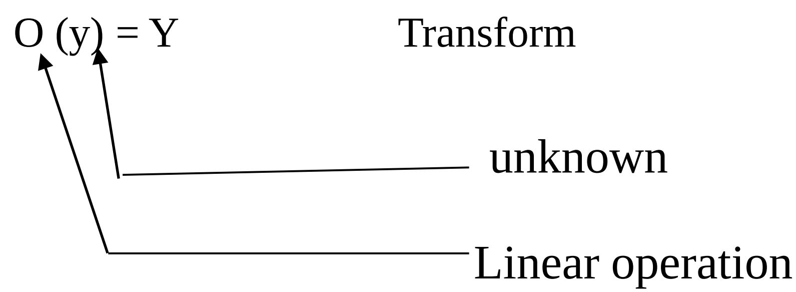
Pierre Simmon Marquis De Laplace (1749-1827), a French Mathematician introduced Laplace Transformations.

Laplace transformation is a technique for solving differential equations.

Or **Laplace Transformations** is a powerful Technique; it **replaces operations of calculus by operations of Algebra.**

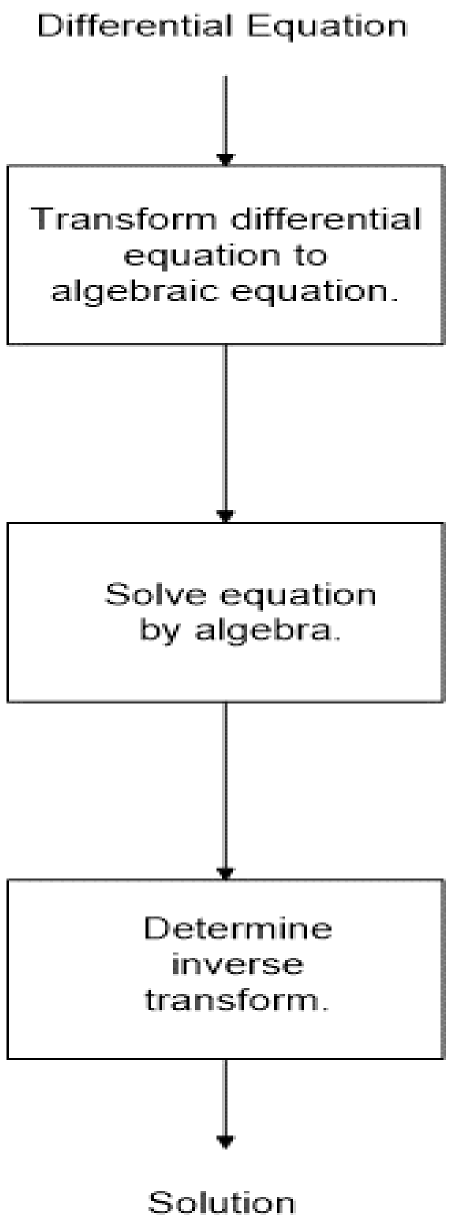
General Transformation

For e.g. With the application of L.T to an Initial value problem, consisting of an Ordinary(or Partial) differential equation (O.D.E) together with initial conditions is reduced to a problem of solving an algebraic equation (with any given Initial conditions automatically taken care)



$O^{-1}(Y) = y$ Inverse transform

In this chapter we use the Laplace transform to convert a problem for an unknown function *f* into a simpler problem for *F*, solve for *F*, and then recover *f* from its transform *F*.



Definition of Laplace transform

Let $f(t)$ be a function defined for $t \geq 0$, and satisfies certain conditions to be named later.
The **Laplace Transform of *f*** is defined as

$$L\{f(t)\} = F(s) = \int_0^{\infty} e^{-st} f(t)dt$$

Here, L is called Laplace Transform operator. The function *f(t)* is known as determining function, depends on *t*. The new function which is to be determined F(s) is called generating function, depends on *s*.

Note here question will be in t and **answer will be in s**.

$$F(s) = \int_0^{\infty} f(t)e^{-st}dt$$

For example the Laplace transform of $f(t) = 2$ for $t \geq 0$ is:

$$L\{f(t)\} = \int_{t=0}^{\infty} e^{-st} f(t)dt$$

$$= \int_{t=0}^{\infty} e^{-st} 2dt$$

$$= 2\left[\frac{e^{-st}}{-s}\right]_{t=0}^{\infty}$$

$$= 2(0-(-1/s)) = \frac{2}{s}$$

Laplace transforms of common functions

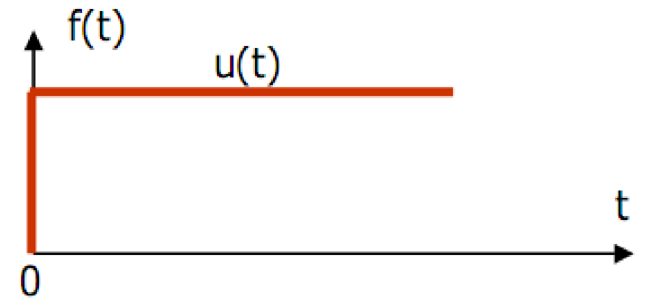
1. Laplace Transformation of constant $C = \frac{C}{s}$

$$L\{f(t)\} = \int_0^\infty e^{-st} f(t)dt = F(s)$$

$$\rightarrow L\{C\} = \int_0^\infty e^{-st}\, C\, dt = F(s)$$

$$\rightarrow L\{C\} = C\int_0^\infty e^{-st}\, dt = F(s)$$

$$\rightarrow L\{C\} = C\left[\frac{e^{-st}}{-s}\right]_0^\infty = \frac{C}{s}$$



Example find Laplace of 1 when $t \geq 0$

$$L(f) = L(1) = \int_0^\infty e^{-st}dt = -\frac{1}{s}e^{-st}\Big|_0^\infty = \frac{1}{s}$$

2. Laplace Transformation of Exponential e^{-at}

$$\mathbf{L\{f(t)\} = \int_0^\infty e^{-st}\, e^{-at}\, dt = F(s)}$$

$$L\{f(t)\} = \int_0^\infty e^{-st} f(t)dt = F(s)$$

$$L\{f(t)\} = \int_0^\infty e^{-(a+s)t}\, dt = F(s)$$

$$L\{f(t)\} = \left[\frac{e^{-(a+s)t}}{-(s+a)}\right]_0^\infty = \frac{1}{s+a}$$

Example

$$L\{e^{-4t}\} = \frac{1}{s+4}$$

Exponential function

$$f(t)=e^{at}$$

$$L[f(t)u(t)]=\int_0^{\infty} e^{at}e^{-st}dt=\int_0^{\infty} e^{-(s-a)t}dt=\left[\frac{-e^{-(s-a)t}}{(s-a)}\right]_0^{\infty}=\frac{1}{s-a}$$

Example

$$\mathrm{L}\{e^{3t}\}=\frac{1}{s-3}$$

$$\mathrm{L}\{f(t)\}=\int_0^{\infty} e^{-st}f(t)dt=F(s)$$

Linear function

$$f(t)=t$$

Per partes integration

$$L[f(t)u(t)]=\int_0^{\infty} te^{-st}dt$$

$$\int uv'=uv-\int u'v$$

$$u=t;\quad u'=1$$

$$v'=e^{-st};\quad v=\frac{-e^{-st}}{s}$$

$$L[f(t)u(t)]=\int_0^{\infty} te^{-st}dt=\left[\frac{-t\,e^{-st}}{s}\right]_0^{\infty}-\int_0^{\infty}\frac{1\cdot(-e^{-st})}{s}dt=0+\left[\frac{-e^{-st}}{s^2}\right]_0^{\infty}=\frac{1}{s^2}$$

$$t\triangleq\frac{1}{s^2}$$

Square function

$$f(t)=t^2$$

Per partes integration

$$L[f(t)u(t)]=\int_0^{\infty} t^2e^{-st}dt$$

$$\int uv'=uv-\int u'v$$

$$u=t^2;\quad u'=2t$$

$$v'=e^{-st};\quad v=\frac{-e^{-st}}{s}$$

$$\mathrm{L}\{f(t)\}=\int_0^{\infty} e^{-st}f(t)dt=F(s)$$

$$L[f(t)u(t)]=\int_0^{\infty} t^2e^{-st}dt=\left[\frac{-t^2\,e^{-st}}{s}\right]_0^{\infty}-\int_0^{\infty}\frac{-2te^{-st}}{s}dt=0+\frac{2}{s}\int_0^{\infty} te^{-st}dt=\frac{2}{s}\frac{1}{s^2}$$

$$t^2\triangleq\frac{2}{s^3}$$

The n-th power function

$$f(t)=t^n$$

$$t^n\triangleq\frac{n!}{s^{n+1}}$$

Laplace Transformation of t^n

We know that $L\{f(t)\} = \int_0^{\infty} e^{-st} f(t)\, dt = F(s)$

$$\Rightarrow L\{f(t)\} = \int_0^{\infty} e^{-st} t^n \, dt$$

$$Put\ st = x \Rightarrow t = \frac{x}{s}$$

$$dt = \frac{1}{s} dx$$

$$As\ t \to 0\ to\ \infty \quad \Rightarrow x \to 0\ to\ \infty$$

$$\Rightarrow L\{t^n\} = \int_{x=0}^{\infty} e^{-x} \left(\frac{x}{s}\right)^n \frac{1}{s} dx$$

$$\Rightarrow L\{t^n\} = \frac{1}{s^{n+1}} \int_0^{\infty} e^{-x} x^n \, dx$$

$$= \frac{1}{s^{n+1}} \Gamma(n+1) \qquad \left[\because \Gamma(n) = \int_0^{\infty} e^{-x} x^{n-1}\, dt, n \geq 0\right]$$

$$L\{t^n)\} = \frac{n!}{s^{n+1}} \qquad [\because\ \Gamma(n+1) = n!]$$

Definition of Gamma function

$$\Gamma(n) = \int_0^{\infty} e^{-t} t^{n-1}\, dt, n \geq 0$$

(OR)

$$\Gamma(n) = \int_0^{\infty} e^{-x} x^{n-1}\, dt, n \geq 0.$$

Note: i) $\Gamma(n+1) = n\Gamma(n) = n!$

ii) $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$

Example 1 find Laplace transform of $t^{\frac{1}{2}}$

$$L\{t^n)\} = \frac{n!}{s^{n+1}}$$

$$L\left\{t^{\frac{1}{2}})\right\} = \frac{\frac{1}{2}!}{s^{\frac{1}{2}+1}}$$

$$= \frac{1}{s^{\frac{1}{2}+1}} \Gamma\left(\frac{3}{2}\right) = \frac{1}{s^{\frac{3}{2}}} \frac{\sqrt{\pi}}{2} \qquad [\because\ n\Gamma(n) = n!\ ]$$

Example 2 find Laplace transform of $t^{-\frac{1}{2}}$

$$L\{t^n)\} = \frac{n!}{s^{n+1}}$$

$$L\left\{t^{-\frac{1}{2}})\right\} = \frac{-\frac{1}{2}!}{s^{-\frac{1}{2}+1}}$$

Note: i) $\Gamma(n+1) = n\Gamma(n) = n!$

ii) $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$

$$= \frac{\Gamma\left(-\frac{1}{2}+1\right)}{s^{\frac{1}{2}}} \Rightarrow L\left(t^{-\frac{1}{2}}\right) = \frac{\Gamma\left(\frac{1}{2}\right)}{\sqrt{s}} = \sqrt{\frac{\pi}{s}}$$

$$L\{f(t)\} = \int_0^\infty e^{-st} f(t)dt = F(s)$$

Laplace transform of *cos (ωt)*

$$L[\cos(wt)] = \int_0^\infty \frac{(e^{jwt} + e^{-jwt})}{2} e^{-st} dt$$

$$= \frac{1}{2}\left[\frac{1}{s-jw} - \frac{1}{s+jw}\right]$$

$$= \frac{s}{s^2+w^2}$$

Laplace transform of *sin (ωt)*

$$L[\sin(\omega t)] = \int_0^\infty \frac{(e^{j\omega t} - e^{-j\omega t})}{2j} e^{-st} dt = \frac{1}{2j}\left[\int_0^\infty e^{-(s-j\omega)t} dt - \int_0^\infty e^{-(s+j\omega)t} dt\right]$$

$$= \frac{1}{2j}\left[\frac{1}{s-j\omega} - \frac{1}{s+j\omega}\right]$$

$$= \frac{\omega}{s^2+\omega^2}$$

Example find

$$L(cos(7t)) = \frac{s}{s^2+49}$$

Example find

$$L(\sin(6t)) = \frac{6}{s^2+36}$$

Laplace transformation of hyperbolic

Laplace transformation of cosh at

$L\{\cosh at\}$

Solution: since $\cosh at = \dfrac{e^{at} + e^{-at}}{2}$

$$L\{f(t)\} = \int_0^{\infty} e^{-st} f(t)dt = F(s)$$

$$\text{Now, } L\{\cosh at\} = \frac{1}{2}[L\{e^{at} + e^{-at}\}]$$

$$= \frac{1}{2}[L(e^{at}) + L(e^{-at})]$$

$$= \frac{1}{2}\left[\frac{1}{s-a} + \frac{1}{s+a}\right]$$

$$\therefore L[\cosh at] = \frac{s}{s^2 - a^2}$$

Laplace transformation of sinh at

$L\{\sinh at\}$

Solution: since $\sinh at = \dfrac{e^{at} - e^{-at}}{2}$

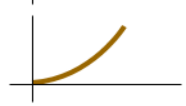
$$L\{f(t)\} = \int_0^{\infty} e^{-st} f(t)dt = F(s)$$

$$\text{Now, } L\{\sinh at\} = \frac{1}{2}[L\{e^{at} - e^{-at}\}]$$

$$= \frac{1}{2}[L(e^{at}) - L(e^{-at})]$$

$$= \frac{1}{2}\left[\frac{1}{s-a} - \frac{1}{s+a}\right]$$

$$\therefore L[\sinh at] = \frac{a}{s^2 - a^2}$$

Name	$f(t)$		$F(s)$
Step	$f(t) = 1$		$\frac{1}{s}$
Ramp	$f(t) = t$		$\frac{1}{s^2}$
Exponential	$f(t) = e^{at}$		$\frac{1}{s-a}$

First shaft property

Statement: If $L\{f(t)\} = F(s)$ then $L\{e^{at}f(t)\} = F(s-a)$

proof

$L\{f(t)\} = \int_0^{\infty} e^{-st} f(t)dt = F(s)$

$$L[e^{-at} f(t)] = \int_0^{\infty} [e^{-at} f(t)]e^{-st} dt$$

$$= \int_0^{\infty} f(t)e^{-(s+a)t} dt = F(s+a)$$

$$L[e^{-at} f(t)] = F(s+a)$$

- whenever we want to evaluate $L\{e^{at}f(t)\}$, first evaluate $L\{f(t)\}$ which is equal to $F(s)$ and then evaluate $L\{e^{at}f(t)\}$, which will be obtained simply, by substituting $s-a$ in place of a in $F(s)$.

Example : find the Laplace of e^{-at} cos (ωt) using shaft property

solution

In this case, *f(t) = cos(ωt)* so,

$$F(s) = \frac{s}{s^2 + w^2}$$

$$\text{and } F(s+a) = \frac{(s+a)}{(s+a)^2 + w^2}$$

$$L[e^{-at}\cos(\omega t)] = \frac{(s+a)}{(s+a)^2 + (\omega)^2}$$

Laplace transformations of derivatives

Statement: If $L\{f(t)\} = F(s)$, then $L\{f'(t)\} = sF(s) - f(0)$

$$L\{f''(t)\} = s^2F(s) - sf(0) - f'(0)$$

$$L\{f^{(n)}(t)\} = s^nF(s) - s^{n-1}f(0) - s^{n-2}f'(0) - \dots - f^{n-1}(0)$$

If the L[f(t)] = F(s), we want to show:

$$L[\frac{df(t)}{dt}] = sF(s) - f(0)$$

Integrate by parts:

$$u = e^{-st},\ du = -se^{-st}dt \text{ and}$$

$$dv = \frac{df(t)}{dt}dt = df(t), \text{ so } v = f(t)$$

$$L\{f(t)\} = \int_0^{\infty} e^{-st} f(t)dt = F(s)$$

Making the previous substitutions gives,

$$L\left[\frac{df}{dt}\right] = f(t)e^{-st}\Big|_0^{\infty} - \int_0^{\infty} f(t)\left[-se^{-st}\right]dt$$

$$= 0 - f(0) + s\int_0^{\infty} f(t)e^{-st}dt$$

$$L\left[\frac{df(t)}{dt}\right] = sF(s) - f(0)$$

We can extend the previous to show:

$$L\left[\frac{df(t)^2}{dt^2}\right]=s^2F(s)-sf(0)-f'(0)$$

$$L\left[\frac{df(t)^3}{dt^3}\right]=s^3F(s)-s^2f(0)-sf'(0)-f''(0)$$

general case

$$L\left[\frac{df(t)^n}{dt^n}\right]=s^nF(s)-s^{n-1}f(0)-s^{n-2}f'(0)$$

$$-\ldots-f^{(n-1)}(0)$$

Examples **2^{nd} derivative**

Example: $f(t)=t^2 \quad L\{t^2\}=?$

$f(0)=0,\ f'(0)=0,\ f''(0)=2$

$L\{f''\}=s^2L\{f\}-s\underbrace{f(0)}_{0}-\underbrace{f'(0)}_{0}$

$L\{2\}=2L\{1\}=\frac{2}{s}=s^2L\{t^2\}\ \therefore L\{t^2\}=\frac{2}{s^3}$

Ex: $L\{\sin 3t\}\ f(t)=\sin 3t$

$f'(t)=3\cos 3t$

$f''(t)=-9\sin 3t$

$L\{-9\sin 3t\}=s^2L\{\sin 3t\}-3$

$(s^2+9)L\{\sin 3t\}=3 \quad \therefore L\{\sin 3t\}=\frac{3}{s^2+9}$

Transform of integrals

Time integration $L[\int_0^t f(x)dx\,]$

$$L[\int_0^t f(x)dx\,] = \int_0^\infty \left[\int_0^t f(x)dx\right] e^{-st}dt$$

Per partes integration

$$\int uv' = uv - \int u'v$$

$$u = \int_0^t f(x)dx\,; \quad u' = f(t)$$

$$v' = e^{-st}; \quad v = -\frac{1}{s}e^{-st}$$

$$L\left[\int_0^\infty f(t)dt\right] = \frac{1}{s}\int_0^\infty f(t)e^{-st}dt$$

$$= \frac{1}{s}F(s)$$

$$L[\int_0^t f(x)dx\,] = [-\frac{1}{s}e^{-st}\int_0^t f(x)dx\,]_0^\infty - \int_0^\infty [-\frac{1}{s}e^{-st}]f(t)dt = \frac{1}{s}\int_0^\infty f(t)e^{-st}dt$$

$$L[\int_0^t f(x)dx\,] = \frac{1}{s}F(s)$$

$$L\{f(t)\} = \int_0^\infty e^{-st} f(t)dt = F(s)$$

Example

Find $\mathcal{L}\left\{\int_0^t e^{-x}\cos x\,dx\right\}$.

$$L[e^{-at}\cos(\omega t)] = \frac{(s+a)}{(s+a)^2 + (\omega)^2}$$

solution

$$\frac{s+1}{s[(s+1)^2+1]}$$

Multiplication of ‘ t’

Theorem

If L$L\{f(t) = F(s)\}$, then $L\left\{t.f(t) = -\frac{d}{ds}F(s)\right\}$

Since

$$L\{f(t)\} = \int_0^\infty e^{-st} f(t)dt = F(s)$$

$$\frac{d}{ds}F(s) = \frac{d}{ds}\left[\int_0^\infty e^{-st} f(t)dt\right]$$

$$= \int_0^\infty f(t)\frac{\partial}{\partial s}(e^{-st})\, dt$$

$$= \int_0^\infty f(t)\,(e^{-st})\,(-t)\, dt$$

$$= -\int_0^\infty f(t)\,(e^{-st})\, t\, dt$$

$$= -L\{t\, f(t)\}$$

$$\therefore \boldsymbol{L\{t.f(t)\} = -\frac{dF}{ds}}$$

$$L\{t^n.f(t)\} = -\frac{d^n}{ds^n}[F(s)] = -\frac{d^nF}{ds^n}$$

Division by ‘ t’ or Laplace integrals

Theorem

If $L\{f(t) = F(s)\}$, then $L\left\{\frac{f(t)}{t}\right\} = \int_s^\infty F(s)$

$$L\{f(t)\} = \int_0^\infty e^{-st} f(t)dt = F(s)$$

Integrate both sides

$$\int_s^\infty F(s)ds = \int_{s=s}^\infty\left(\int_{t=0}^\infty e^{-st} f(t)dt\right).ds$$

Since “s” and “t” are independent variables, by interchanging the order of integration

$$\int_s^\infty F(s)ds = \int_{t=0}^\infty\left(\int_{s=s}^\infty e^{-st}\, ds\right).f(t)dt$$

$$\int_s^\infty F(s)ds = \int_{s=0}^\infty [e^{-st}]_s^\infty.\frac{1}{-t}\, f(t)dt$$

$$= \int_{t=0}^\infty e^{-st}\left(\frac{f(t)}{t}\right) dt$$

$$\Rightarrow \int_s^\infty F(s)\, ds = L\left\{\frac{f(t)}{t}\right\}$$

$$\therefore L\left\{\frac{f(t)}{t}\right\} = \int_s^\infty F(s)\, ds$$

Example: $find\ Laplace\ transform\ of\ \ t^2e^{4t}$

L(t^2e^{4t})

L(e^{4t}) = $\frac{1}{s-4}$

$L\{t^n.f(t)\} = -\frac{d^n}{ds^n}[F(s)] = -\frac{d^nF}{ds^n}$

L(t^2e^{4t})= - $\frac{d}{ds}(\frac{1}{s-4})$ = $\frac{1}{(s-4)^2}$

L(t^2e^{4t})= - $\frac{d^2}{ds^2}(\frac{1}{s-4})$ = $\frac{2}{(s-4)^3}$

$$L[e^{-at}f(t)] = F(s+a)$$

Example

$$t^n \triangleq \frac{n!}{s^{n+1}}$$

- Find $L\{t\ \sin at\}$

Sol: we know that $L\{t.f(t)\} = -\frac{dF}{ds}$

$Here\ f(t) = \sin at$

$\Longrightarrow F(s) = \left[\frac{a}{s^2+a^2}\right]$

$\therefore L\{t\ \sin at\} = -\frac{d}{ds}\left\{\frac{a}{s^2+a^2}\right\} = \frac{2as}{(s^2+a^2)^2}$

Linearity

Linearity of Laplace transform

$$L[c_1 f_1(t) + c_2 f_2(t)] = c_1 F_1(s) + c_2 F_2(s)$$

Example of function *f* :

$$f(t) = 5\,e^{-2t} - 3\sin(4t).$$

Laplace transform by linearity: we find

$$\text{L}(f(t)) = 5\,L(e^{-2t}) - 3\,L(\sin(4t)).$$

$$= \frac{5}{s+2} - \frac{12}{s^2+16.}$$

As an another example, by property)

$$\text{L}(5\,e^{5t} + \cos(4t))$$

$$= \text{L}(5\,e^{5t}) + L(\cos(4t)) = \frac{5}{s-5} + \frac{s}{s^2+16}$$

An example where both (1) and (2) are used,

$$L(3\,t^7 + 8) = L(3\,t^7 + L(\,8) = 3\left(\frac{7!}{s^8}\right) + 8\left(\frac{1}{s}\right)$$

As an example, we determine

$$\begin{aligned}\mathcal{L}(3+e^{6t})^2 &= \mathcal{L}(3+e^{6t})(3+e^{6t}) = \mathcal{L}(9+6e^{6t}+e^{12t}) \\ &= \mathcal{L}(9) + \mathcal{L}(6e^{6t} + \mathcal{L}(e^{12t}) \\ &= 9\mathcal{L}(1) + 6\mathcal{L}(e^{6t}) + \mathcal{L}(e^{12t}) \\ &= \frac{9}{s} + \frac{6}{s-6} + \frac{1}{s-12}.\end{aligned}$$

1. Table of Laplace transforms

Original	Image
$f(t)$	$F(s)$
$f(t)e^{at}$	$F(s-a)$
$f(at)$	$\frac{1}{a}F(\frac{s}{a})$
$f'(t)$	$sF(s)-f(0+)$
$f''(t)$	$s^2F(s)-sf(0+)-f'(0+)$
$f^{(n)}(t)$	$s^nF(s)-s^{n-1}f(0+)-s^{n-2}f'(0+)-\ldots-f^{(n-1)}(0+)$
$tf(t)$	$-F'(s)$
$t^nf(t)$	$(-1)^nF^{(n)}(s)$
$\int_0^t f(x)dx$	$\frac{1}{s}F(s)$

The Laplace transform

The most commonly used transform pairs

Original	Image
a	$\frac{a}{s}$
t	$\frac{1}{s^2}$
t^2	$\frac{2}{s^3}$
$t^n, n \in N$	$\frac{n!}{s^{n+1}}$
e^{at}	$\frac{1}{s-a}$
te^{at}	$\frac{1}{(s-a)^2}$
t^2e^{at}	$\frac{2}{(s-a)^3}$
$t^ne^{at}, n \in N$	$\frac{n!}{(s-a)^{n+1}}$

Original	Image
$\sin(\omega t)$	$\frac{\omega}{s^2+\omega^2}$
$\cos(\omega t)$	$\frac{s}{s^2+\omega^2}$
$\sinh(\omega t)$	$\frac{\omega}{s^2-\omega^2}$
$\cosh(\omega t)$	$\frac{s}{s^2-\omega^2}$
$t\sin(\omega t)$	$\frac{2s\omega}{(s^2+\omega^2)^2}$
$t\cos(\omega t)$	$\frac{s^2-\omega}{(s^2+\omega^2)^2}$
$e^{at}\sin(\omega t)$	$\frac{\omega}{(s-a)^2+\omega^2}$
$e^{at}\cos(\omega t)$	$\frac{s-a}{(s-a)^2+\omega^2}$

Inverse Laplace transformations

$L\{f(t)\} = F(s)$ then *f(t)* is called the inverse Laplace transform of *f(s)* and is denoted by

$L^{-1}\{F(s)\} = f(t)$ $f(t) = L^{-1}\{F(s)\}$

Where L^{-1} is inverse Laplace

Example

We have

$$\mathcal{L}^{-1}\left[\frac{4}{s-3}\right]\Bigg|_t = 4e^{3t}$$

Because

$$\frac{4}{s-3} = \mathcal{L}\left[4e^{3t}\right]$$

Example

$$\mathcal{L}(\sin(6t)) = \frac{6}{s^2+36}.$$

$$\mathcal{L}^{-1}\left(\frac{6}{s^2+36}\right) = \sin(6t)$$

Example find the inverse Laplace transform of $F(s) = \frac{4s+7}{s^2+16}$

Since we know, that $\frac{s}{s^2+\omega^2} \hat{=} \cos\omega t$ *and* $\frac{\omega}{s^2+\omega^2} \hat{=} \sin\omega t,$

it will be helpful to rearrange the original formula $\frac{4s+7}{s^2+16} = 4\frac{s}{s^2+16} + \frac{7}{4}\frac{4}{s^2+16}$

Now we can directly write the result by taking the inverse Laplace $L^{-1}\{F(s)\} = f(t)$

$$f(t) = 4\cos 4t + \frac{7}{4}\sin 4t,\ t \geq 0$$

Example find the inverse Laplace transform of $Y(s) = \frac{4(s-1)}{(s-1)^2+4}$

$$\cos 2t \leftrightarrow \frac{s}{s^2+4} \qquad e^t\cos 2t \leftrightarrow \frac{s-1}{(s-1)^2+4}.$$

Hence,

$$\begin{aligned} y(t) &= \mathcal{L}^{-1}\left\{\frac{4(s-1)}{(s-1)^2+4}\right\} \\ &= 4\,\mathcal{L}^{-1}\left\{\frac{s-1}{(s-1)^2+4}\right\} \\ &= 4e^t\cos 2t. \end{aligned}$$

Example find

$$\mathcal{L}^{-1}\left(\frac{5}{s^{11}}\right) = 5\mathcal{L}^{-1}\left(\frac{1}{s^{11}}\right)$$

$$= \frac{5}{10!}\mathcal{L}^{-1}\left(\frac{10!}{s^{11}}\right)$$

$$= \frac{5}{10!}t^{10}.$$

Example:- find the inverse Laplace transform of $\frac{30}{s^7} + \frac{8}{s-4}$

We know $\mathcal{L}^{-1}\left[\frac{6!}{s^7}\right]\Big|_t = t^6$ *and* $\mathcal{L}^{-1}\left[\frac{1}{s-4}\right]\Big|_t = e^{4t}$

So,

$$\mathcal{L}^{-1}\left[\frac{30}{s^7} + \frac{8}{s-4}\right]\Big|_t = \frac{30}{6!}\mathcal{L}^{-1}\left[\frac{6!}{s^7}\right]\Big|_t + 8e^{4t} = \frac{30}{6\cdot5\cdot4\cdot3\cdot2}t^6 + 8e^{4t}$$

$$= \frac{1}{24}t^6 + 8e^{4t}$$

Example

Find: $\mathcal{L}^{-1}\left(\frac{4s+1}{s^2+10s+34}\right)$.

Here the denominator does not factor over the reals.

Hence complete the square.

$$s^2+10s+34=\underbrace{s^2+10s+25}-25+34=(s+5)^2+9.$$

$$\mathcal{L}^{-1}\left(\frac{4s+1}{s^2+10s+34}\right)=\mathcal{L}^{-1}\left(\frac{4s+1}{(s+5)^2+9}\right)$$

this s must now be made into (s+5).

$$=\mathcal{L}^{-1}\left(\frac{4(s+5)-20+1}{(s+5)^2+9}\right)=\mathcal{L}^{-1}\left(\frac{4(s+5)-19}{(s+5)^2+9}\right)$$

$$=4\mathcal{L}^{-1}\left(\frac{(s+5)}{(s+5)^2+9}\right)-19\mathcal{L}^{-1}\left(\frac{1}{(s+5)^2+9}\right)$$

$$=4e^{-5t}\cos(3t)-\frac{19}{3}\mathcal{L}^{-1}\left(\frac{3}{(s+5)^2+9}\right)$$

$$=4e^{-5t}\cos(3t)-\frac{19}{3}e^{-5t}\sin(3t).$$

$e^{at}\sin(\omega t)$	$\frac{\omega}{(s-a)^2+\omega^2}$
$e^{at}\cos(\omega t)$	$\frac{s-a}{(s-a)^2+\omega^2}$

PARTIAL FRACTION EXPANSION

Definition -- Partial fractions are several fractions whose sum equals a given fraction Purpose -- Working with transforms requires breaking complex fractions into simpler fractions to allow use of tables of transforms

Case 1 : If the denominator has non-repeated linear factors $(s-a),(s-b),(s-c)$, then

$$\frac{f(s)}{(s-a)(s-b)(s-c)}=\frac{A}{(s-a)}+\frac{B}{(s-b)}+\frac{C}{(s-c)}$$

Case 2 : If the denominator has repeated linear factors $(s-a)$, (n times), then

$$\frac{f(s)}{(s-a)^n}=\frac{A_1}{(s-a)}+\frac{A_2}{(s-a)^2}+\frac{A_3}{(s-a)^3}+\cdots+\frac{A_n}{(s-a)^n}$$

Case 3 : If the denominator has non-repeated quadratic factors (s^2+as+b), (s^2+cs+d),

$$\frac{f(s)}{(s^2+as+b)(s^2+cs+d)}=\frac{As+B}{(s^2+as+b)}+\frac{Cs+D}{(s^2+cs+d)}$$

Case 4 : If the denominator has repeated quadratic factors (s^2+as+b), (n times), then

$$\frac{f(s)}{(s^2+as+b)^n}=\frac{As+B}{(s^2+as+b)}+\frac{Cs+D}{(s^2+as+b)^2}+\cdots\text{(n times)}$$

Example find the inverse Laplace transform of $$\frac{3s+5}{s^2-3s-10} = \frac{3s+5}{(s-5)(s+2)}$$

We are looking for coefficients *A, B* and *C* $$\frac{3s+5}{(s-5)(s+2)} = \frac{A}{s-5} + \frac{B}{s+2}.$$

To determine *A* and *B*, first clear the denominators:

$$\frac{3s+5}{(s-5)(s+2)}(s-5)(s+2) = \frac{A}{(s-5)}(s-5)(s+2) + \frac{B}{(s+2)}(s-5)(s+2).$$

We get $$3s+5 = A(s+2) + B(s-5) = (A+B)s + 2A - 5B.$$

By comparing the coefficients of *s* and constant coefficients, we get two equations in *A* and *B*.

$$\begin{aligned} A + B &= 3 \\ 2A - 5B &= 5 \end{aligned}$$

Hence, $$A = \frac{20}{7}, \quad \text{and} \quad B = \frac{1}{7}.$$

We can now determine the inverse transform

$$\begin{aligned} \mathcal{L}^{-1}\left(\frac{3s+5}{(s-5)(s+2)}\right) &= \mathcal{L}^{-1}\left(\frac{A}{s-5} + \frac{B}{s+2}\right) \\ &= A\,\mathcal{L}^{-1}\left(\frac{1}{s-5}\right) + B\,\mathcal{L}^{-1}\left(\frac{1}{s+2}\right) = \frac{20}{7}e^{5t} + \frac{1}{7}e^{-2t}. \end{aligned}$$

Example find the inverse Laplace transform of $$F(s)=\frac{2s+3}{s^2+4s+3}$$

We have to find coefficients *A* and *B* for $$F(s)=\frac{2s+3}{s^2+4s+3}=\frac{A}{s+3}+\frac{B}{s+1}$$

Multiplying the equation by its denominator $$2s+3=A(s+1)+B(s+3)$$

Now we can substitute

$$s=-3:\quad -3=-2A\Rightarrow A=\frac{3}{2}$$

$$s=-1:\quad 1=2B\Rightarrow B=\frac{1}{2}$$

Therefore *F*(s) $$F(s)=\frac{3}{2}\frac{1}{s+3}+\frac{1}{2}\frac{1}{s+1}$$

Inverse Laplace of *F(s)* $$f(t)=\frac{3}{2}e^{-3t}+\frac{1}{2}e^{-t},\, t\geq 0$$

Example find the inverse Laplace transform of

$$F(s) = \frac{3s^2+5}{(s+1)(s+3)^2}$$

We are looking for coefficients *A, B* and *C*

$$F(s) = \frac{3s^2+5}{(s+1)(s+3)^2} = \frac{A}{s+1} + \frac{B}{(s+3)^2} + \frac{C}{s+3}$$

Multiplying the equation by its denominator

$$3s^2+5 = A(s+3)^2 + B(s+1) + C(s+1)(s+3)$$

Now we can substitute to get *A*,*B*; the *C* coefficient can be obtained by comparison of s^2 factors

$$s=-1: \quad 8 = 4A \Rightarrow A = 2$$

$$s=-3: \quad 32 = -2B \Rightarrow B = -16$$

$$s^2: \quad 3 = A + C \Rightarrow C = 3 - A = 1$$

Therefore *F*(s)

$$F(s) = \frac{2}{s+1} - \frac{16}{(s+3)^2} + \frac{1}{s+3}$$

Inverse Laplace of *F(s)*

$$f(t) = 2e^{-t} - 16\,te^{-3t} + e^{-3t}, t \geq 0$$

Example find the inverse Laplace transform of $F(s)=\frac{2s-7}{(s+6)(s^2+4)}$

We are looking for coefficients *A, B* and *C*

$$\frac{2s-7}{(s+6)(s^2+4)}=\frac{A}{s+6}+\frac{Bs+C}{s^2+4}$$

Multiplying the equation by its denominator

$$2s-7=A(s^2+4)+(Bs+C)(s+6)$$

Now we can substitute to get *A*,*B*; the *C* coefficient can be obtained by comparison of s^2 factors

$$s=-6:\quad -19=40A\Rightarrow A=-\frac{19}{40}$$

$$s^2:\quad 0=A+B\Rightarrow B=-A=\frac{19}{40}$$

$$s^0:\quad -7=4A+6C\Rightarrow C=\frac{1}{6}(-7+\frac{19}{10})=-\frac{51}{60}$$

Therefore *F*(s)

$$F(s)=-\frac{19}{40}\frac{1}{s+6}+\frac{19}{40}\frac{s}{s^2+4}-\frac{51}{120}\frac{2}{s^2+4}$$

Inverse Laplace of *F(s)*

$$f(t)=-\frac{19}{40}e^{-6t}+\frac{19}{40}\cos 2t-\frac{51}{120}\sin 2t,\ t\geq 0$$

Applications of D.E's by using Laplace and inverse Laplace transformations

Suppose the given D.Eq is of the form $a\,\frac{d^2y}{dt^2} + b\,\frac{dy}{dt} + y = f(t) \longrightarrow 1$

is a Linear D.Eq of order 2 with constants a, b the boundary conditions are $y(0) = y'(0) = 0$

We Know that $L\{f^{(n)}(t)\} = s^n F(s) - s^{n-1} f(0) - s^{n-2} f'(0) - \; \ldots . - f^{n-1}(0)$

For *the first derivative* $L\{f'(t)\} = sF(s) - f(0) \Rightarrow L[y'(t)] = s\bar{y}(s) - y(0)$

For the second derivative $L\,[y^2\,(t)] = s^2 L\{y\} - sy(0) - y'(0)$

By simplify

$L\{y(t)\} = y(s)$ *and* $y^2\,(t)$ *is the second derivative*

[Step 1] Taking the Laplace transform of the equation (1)

i.e. $a\,L\{y''\} + \; b\,L\{y'\} + \; L\{y\} = L\{f(t)\}$

$$\Longrightarrow a\{s^2\bar{y}(s) - s\,y(0) - y'(0)\} + b\,\{s\bar{y}(s) - y(0)\} + \bar{y}(s) = F(s)$$

Substituting in the initial conditions, we obtain

$$\Longrightarrow as^2\bar{y}(s) + bs\,\bar{y}(s) + \bar{y}(s) = F(s) \qquad \Longrightarrow (as^2 + bs\; + 1)\,\bar{y}(s) = F(s)$$

[Step 2] Simplify to find Y(s) = L{y} $\Longrightarrow \bar{y}(s) = \dfrac{F(s)}{(as^2 + bs\; + 1)}$

[Step 3] Find the inverse transform $\Longrightarrow y(t) = L^{-1}\{\bar{y}(s)\} = L^{-1}\left\{\frac{F(s)}{(as^2+bs\,+1)}\right\}$

Example: solve the following equation using Laplace transformation

$$y'' - 6y' + 5y = 0, \qquad y(0) = 1, \quad y'(0) = -3$$

[Step 1] Transform both sides $\mathcal{L}\{y'' - 6y' + 5y\} = \mathcal{L}\{0\}$

$$(s^2\mathcal{L}\{y\} - s\,y(0) - y'(0)) - 6(s\mathcal{L}\{y\} - y(0)) + 5\mathcal{L}\{y\} = 0$$

[Step 2] Simplify to find Y(s) = L{y}

$$(s^2\mathcal{L}\{y\} - s - (-3)) - 6(s\,\mathcal{L}\{y\} - 1) + 5\mathcal{L}\{y\} = 0$$

$$(s^2 - 6s + 5)\mathcal{L}\{y\} - s + 9 = 0$$

$$(s^2 - 6s + 5)\mathcal{L}\{y\} = s - 9 \qquad \mathcal{L}\{y\} = \frac{s-9}{s^2 - 6s + 5}$$

[Step 3] Find the inverse transform y(t) Use partial fractions to simplify,

$$\mathcal{L}\{y\} = \frac{s-9}{s^2 - 6s + 5} = \frac{a}{s-1} + \frac{b}{s-5} \qquad s - 9 = a(s-5) + b(s-1) = (a+b)s + (-5a - b)$$

Equating the corresponding coefficients:

$1 = a + b$ $\qquad a = 2$

$-9 = -5a - b$ $\qquad b = -1$

Hence, $\mathcal{L}\{y\} = \frac{s-9}{s^2 - 6s + 5} = \frac{2}{s-1} - \frac{1}{s-5}$. Inverse Laplace $y(t) = 2e^{t} - e^{5t}$.

Example: solve the following equation using Laplace transformation

$$y'' - 3y' + 2y = e^{3t}, \qquad y(0) = 1, \quad y'(0) = 0$$

[Step 1] Transform both sides $(s^2\mathcal{L}\{y\} - s\,y(0) - y'(0)) - 3(s\mathcal{L}\{y\} - y(0)) + 2\mathcal{L}\{y\} = \mathcal{L}\{e^{3t}\}$

[Step 2] Simplify to find *Y*(*s*) = **L**{*y*}

$$(s^2\mathcal{L}\{y\} - s - 0) - 3(s\mathcal{L}\{y\} - 1) + 2\mathcal{L}\{y\} = 1/(s-3)$$

$$(s^2 - 3s + 2)\mathcal{L}\{y\} - s + 3 = 1/(s-3)$$

$$(s^2 - 3s + 2)\mathcal{L}\{y\} = s - 3 + \frac{1}{s-3} = \frac{(s-3)^2 + 1}{s-3}$$

$$\mathcal{L}\{y\} = \frac{s^2 - 6s + 10}{(s^2 - 3s + 2)(s-3)} = \frac{s^2 - 6s + 10}{(s-1)(s-2)(s-3)}$$

[Step 3] Find the inverse transform *y*(*t*) by partial fractions,

$$\mathcal{L}\{y\} = \frac{s^2 - 6s + 10}{(s-1)(s-2)(s-3)} = \frac{5}{2}\frac{1}{s-1} - 2\frac{1}{s-2} + \frac{1}{2}\frac{1}{s-3}.$$

Therefore, $y(t) = \frac{5}{2}e^{t} - 2e^{2t} + \frac{1}{2}e^{3t}.$

Example solve the following equation using Laplace transformation

$$y'' + y = \sin 2t, \quad y(0) = 2, \; y'(0) = 1$$

Solution

Taking the Laplace transform of the equation $\left[s^2 L\{y\} - sy(0) - y'(0)\right] + L\{y\} = 2/(s^2+4)$

Letting $Y(s) = L\{y\}$, we have $\left(s^2+1\right)Y(s) - sy(0) - y'(0) = 2/(s^2+4)$

Substituting in the initial conditions, we obtain $\left(s^2+1\right)Y(s) - 2s - 1 = 2/(s^2+4)$

Thus $$Y(s) = \frac{2s^3 + s^2 + 8s + 6}{(s^2+1)(s^2+4)}$$

•

Using partial fractions $$Y(s) = \frac{2s^3 + s^2 + 8s + 6}{(s^2+1)(s^2+4)} = \frac{As+B}{s^2+1} + \frac{Cs+D}{s^2+4}$$

Then $$2s^3 + s^2 + 8s + 6 = (As+B)(s^2+4) + (Cs+D)(s^2+1)$$

$$= (A+C)s^3 + (B+D)s^2 + (4A+C)s + (4B+D)$$

Solving, we obtain $A = 2$, $B = 5/3$, $C = 0$, and $D = -2/3$. Thus

$$Y(s) = \frac{2s}{s^2+1} + \frac{5/3}{s^2+1} - \frac{2/3}{s^2+4}$$

Hence $$y(t) = 2\cos t + \frac{5}{3}\sin t - \frac{1}{3}\sin 2t$$

Solving of differential equations by the Laplace transform

Example solve the following equation using Laplace transformation

$$x' + 4x = \sin 2t, \quad x(0) = 3$$

We know that

$$x'(t) \,\hat{=}\, sX(s) - x(0)$$

$$\sin \omega t \,\hat{=}\, \frac{\omega}{s^2 + \omega^2}$$

Equation in the Laplace form

$$sX(s) - 3 + 4X(s) = \frac{2}{s^2 + 4}$$

$$X(s)(s+4) - 3 = \frac{2}{s^2+4} \quad \Rightarrow \quad X(s) = \frac{3}{s+4} + \frac{2}{(s^2+4)(s+4)}$$

$$\frac{2}{(s^2+4)(s+4)} = \frac{A}{s+4} + \frac{Bs+C}{s^2+4}$$

$$2 = A(s^2+4) + (Bs+C)(s+4)$$

$$s = -4: \quad 2 = 20A \Rightarrow A = \frac{1}{10}$$

$$s^0: \quad 2 = 4A + 4C \quad \Rightarrow \quad C = \frac{2-4A}{4} = \frac{2}{5}$$

$$s^2: \quad 0 = A + B \Rightarrow B = -A = -\frac{1}{10}$$

A formula for the $X(s)$ after the partial fraction decomposition

$$X(s) = \frac{3}{s+4} + \frac{\frac{1}{10}}{s+4} + \frac{-\frac{1}{10}s + \frac{2}{5}}{s^2+4}$$

after some small arrangements

$$X(s) = \frac{\frac{31}{10}}{s+4} - \frac{1}{10}\frac{s}{s^2+4} + \frac{1}{5}\frac{2}{s^2+4}$$

The answer

$$x(t) = \frac{31}{10}e^{-4t} - \frac{1}{10}\cos 2t + \frac{1}{5}\sin 2t, \; t \ge 0$$

Example solve the below equation using Laplace transformation

$$x'' + 4x = 2\cos 2t, \quad x(0) = 0; \quad x'(0) = 4$$

Necessary relations $x'(t) \mathrel{\hat{=}} sX(s) - x(0)$

$$x''(t) \mathrel{\hat{=}} s^2 X(s) - s\,x(0) - x'(0)$$

Equation in the Laplace form $s^2 X(s) - s\cdot 0 - 4 + 4X(s) = \dfrac{2s}{s^2+4}$

$$X(s)(s^2+4) = 4 + \frac{2s}{s^2+4} \quad \Rightarrow \quad X(s) = \frac{4}{s^2+4} + \frac{2s}{(s^2+4)^2}$$

knowing that $t\sin\omega t \mathrel{\hat{=}} \dfrac{2s\omega}{(s^2+\omega^2)^2}$ $\quad x(t) = 2\sin 2t + \dfrac{1}{2} t \sin 2t$

The original function

$$x(t) = \frac{1}{2}(4+t)\sin 2t,\ t \ge 0$$

Example:-consider a problem that related to arises in the motion of a mass attached to a spring with external force, as shown in Figure

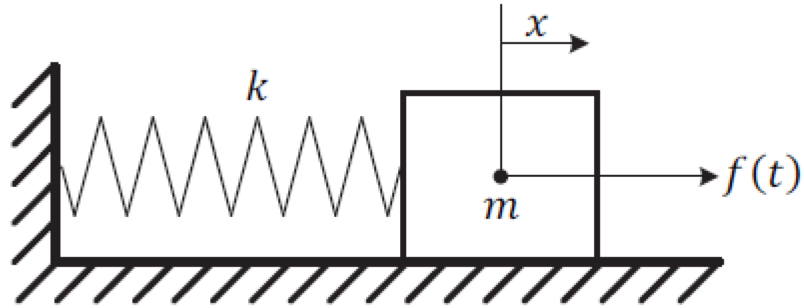


Figure A block-spring system with an external force $f(t)$.

The Equation of motion is

$$mx'' = -kx + f(t)$$

$$x'' = -\frac{k}{m}x + \frac{f(t)}{m}$$

Let $\omega^2 = \frac{k}{m}$ and $A(t) = \frac{f(t)}{m}$. Then, we have

$$x'' + \omega^2 x = A(t)$$

Take $\omega = 2 \; and \; A(t) = \sin 3t$

Use Laplace transform method to solve the following IVP

Solution

The Equation of motion became $\mathcal{L}\{x'' + 4x\} = \mathcal{L}\{\sin 3t\}$

Apply LTs on both sides of the DE

$$\mathcal{L}\{x'' + 4x\} = \mathcal{L}\{\sin 3t\}$$

we get

$$(s^2X(s) - sx(0) - x'(0)) + 4X(s) = \frac{3}{s^2 + 3^2}$$

Plugging in the IC's, we get

$$X(s)(s^2+4)=\frac{3}{s^2+9}$$

$$X(s)=\frac{3}{(s^2+4)(s^2+9)}$$

By partial fraction decomposition

$$\boldsymbol{X(s)=\frac{As+B}{s^2+4}+\frac{Cs+D}{s^2+9}}$$

We get

$$\boldsymbol{B=\frac{3}{5} \qquad D=-\frac{3}{5} \qquad A=0 \qquad C=0}$$

Thus,

$$X(s)=\frac{3}{5}\left(\frac{1}{s^2+4}\right)-\frac{3}{5}\left(\frac{1}{s^2+9}\right)$$

$$=\frac{3}{10}\left(\frac{2}{s^2+2^2}\right)-\frac{1}{5}\left(\frac{3}{s^2+3^2}\right)$$

Finally, applying inverse LT generates the PS to the original DE

$$x(t)=\mathcal{L}^{-1}\{X(s)\}$$

$$=\mathcal{L}^{-1}\left\{\frac{3}{10}\left(\frac{2}{s^2+2^2}\right)-\frac{1}{5}\left(\frac{3}{s^2+3^2}\right)\right\}$$

$$=\frac{3}{10}\mathcal{L}^{-1}\left\{\frac{2}{s^2+2^2}\right\}-\frac{1}{5}\mathcal{L}^{-1}\left\{\frac{3}{s^2+3^2}\right\}$$

$$=\frac{3}{10}\sin 2t-\frac{1}{5}\sin 3t$$